Chapter 1

Warm-up problem set

1.1 Applications

1. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 8.$$

2. If a, b, c are non-negative numbers, then

$$a^3 + b^3 + c^3 - 3abc \geq 2\left(\frac{b+c}{2} - a\right)^3.$$

3. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a+b+c}{3} \geq \sqrt[5]{\frac{a^2+b^2+c^2}{3}}.$$

4. Let a, b, c be non-negative numbers such that $a^3 + b^3 + c^3 = 3$. Prove that

$$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$

(*Vasile Cîrtoaje*, GM-A, 1, 2003)

5. If a, b, c are non-negative numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

(*Darij Grinberg*, MS, 2004)

6. If a, b, c are distinct real numbers, then

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2.$$

7. If a, b, c are non-negative numbers, then

$$(a^2 - bc)\sqrt{b+c} + (b^2 - ca)\sqrt{c+a} + (c^2 - ab)\sqrt{a+b} \geq 0.$$

8. If a, b, c, d are non-negative real numbers, then

$$\frac{a-b}{a+2b+c} + \frac{b-c}{b+2c+d} + \frac{c-d}{c+2d+a} + \frac{d-a}{d+2a+b} \geq 0.$$

9. Let a, b, c be non-negative numbers such that

$$a^2 + b^2 + c^2 = a + b + c.$$

Prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \leq ab + bc + ca.$$

(*Vasile Cîrtoaje*, MS, 2006)

10. Let a, b, c be non-negative numbers, no two of them are zero. Then,

$$\frac{a^2}{a^2+ab+b^2} + \frac{b^2}{b^2+bc+c^2} + \frac{c^2}{c^2+ca+a^2} \geq 1.$$

11. If a, b, c are non-negative numbers, then

$$\sqrt{\frac{a^3}{a^3+(b+c)^3}} + \sqrt{\frac{b^3}{b^3+(c+a)^3}} + \sqrt{\frac{c^3}{c^3+(a+b)^3}} \geq 1.$$

12. Let a, b, c be positive numbers and let

$$E(a,b,c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that:

a) $(a+b+c)E(a,b,c) \geq ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2$;

b) $2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)E(a,b,c) \geq (a-b)^2 + (b-c)^2 + (c-a)^2$.

(*Vasile Cîrtoaje*, MS, 2005)

13. Let a, b, c and x, y, z be real numbers such that $a+x \geq b+y \geq c+z \geq 0$ and $a+b+c = x+y+z$. Prove that

$$ay + bx \geq ac + xz.$$

14. Let $a, b, c \in \left[\frac{1}{3}, 3\right]$. Prove that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \geq \frac{7}{5}.$$

15. Let a, b, c and x, y, z be non-negative numbers such that

$$a + b + c = x + y + z.$$

Prove that

$$ax(a + x) + by(b + y) + cz(c + z) \geq 3(abc + xyz).$$

(*Vasile Cîrtoaje*, MS, 2005)

16. If a, b, c are non-negative numbers, then

$$4(a + b + c)^3 \geq 27(ab^2 + bc^2 + ca^2 + abc).$$

17. Let a, b, c be non-negative numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2ab^2+1} + \frac{1}{2bc^2+1} + \frac{1}{2ca^2+1} \geq 1.$$

18. If a, b, c, d are positive numbers, then

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \geq \frac{4}{ac+bd}.$$

19. If $a, b, c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, then

$$\frac{3}{a+2b} + \frac{3}{b+2c} + \frac{3}{c+2a} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}.$$

20. Let a, b, c be non-negative numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

21. Let a, b, c be non-negative real numbers such that $ab+bc+ca = 3$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \geq \frac{3}{2}.$$

(*Vasile Cîrtoaje*, MS, 2005)

22. Let a, b, c be non-negative numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \le 1.$$

(*Vasile Cîrtoaje*, MS, 2005)

23. Let a, b, c be positive numbers such that $abc = 1$. Prove that

a) $\dfrac{a-1}{b} + \dfrac{b-1}{c} + \dfrac{c-1}{a} \ge 0;$

b) $\dfrac{a-1}{b+c} + \dfrac{b-1}{c+a} + \dfrac{c-1}{a+b} \ge 0.$

24. Let a, b, c, d be non-negative numbers such that $a^2 - ab + b^2 = c^2 - cd + d^2$. Prove that

$$(a+b)(c+d) \ge 2(ab+cd).$$

25. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{1+(n-1)a_1} + \frac{1}{1+(n-1)a_2} + \cdots + \frac{1}{1+(n-1)a_n} \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 10, 1991)

26. Let a, b, c, d be non-negative real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$. Prove that

$$(1-a)(1-b)(1-c)(1-d) \ge abcd.$$

(*Vasile Cîrtoaje*, GM-B, 9-10, 2001)

27. If a, b, c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \le 3.$$

(*Vasile Cîrtoaje*, GM-B, 7-8, 1992)

28. If a, b, c, d are positive real numbers, then

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+d}\right)^2 + \left(\frac{d}{d+a}\right)^2 \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 6, 1995)

29. Let a, b, c be positive numbers such that $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. If $a \le b \le c$, then

$$ab^2c^3 \ge 1.$$

(*Vasile Cîrtoaje*, GM-B, 11, 1998)

30. Let a, b, c be non-negative numbers, no two of them are zero. Then

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \ge \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

(*Vasile Cîrtoaje*, GM-B, 10, 2002)

31. If a, b, c are non-negative numbers, then

$$2(a^2+1)(b^2+1)(c^2+1) \ge (a+1)(b+1)(c+1)(abc+1).$$

(*Vasile Cîrtoaje*, GM-A, 2, 2001)

32. If a, b, c are non-negative numbers, then

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge 1+abc+a^2b^2c^2.$$

(*Vasile Cîrtoaje*, *Mircea Lascu*, RMT, 1-2, 1989)

33. If a, b, c, d are non-negative numbers, then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2) \ge \left(\frac{1+abcd}{2}\right)^2.$$

(*Vasile Cîrtoaje*, GM-B, 1, 1992)

34. If a, b, c are non-negative numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (ab+bc+ca)^3.$$

(*Vasile Cîrtoaje*, *Mircea Lascu*, ONI, 1995)

35. Let a, b, c, d be positive numbers such that $abcd = 1$. Prove that

$$\frac{1}{1+ab+bc+ca}+\frac{1}{1+bc+cd+db}+\frac{1}{1+cd+da+ac}+\frac{1}{1+da+ab+bd} \le 1.$$

36. If a, b, c and x, y, z are real numbers, then

$$4(a^2 + x^2)(b^2 + y^2)(c^2 + z^2) \geq 3(bcx + cay + abz)^2.$$

(*Vasile Cîrtoaje*, MS, 2004)

37. If $a \geq b \geq c \geq d \geq e$, then

$$(a + b + c + d + e)^2 \geq 8(ac + bd + ce).$$

For $e \geq 0$, determine when equality occurs.

(*Vasile Cîrtoaje*, MS, 2005)

38. If a, b, c, d are real numbers, then

$$6(a^2 + b^2 + c^2 + d^2) + (a + b + c + d)^2 \geq 12(ab + bc + cd).$$

(*Vasile Cîrtoaje*, MS, 2005)

39. If a, b, c are positive numbers, then

$$\sqrt{(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \geq 1 + \sqrt{1 + \sqrt{(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}}.$$

(*Vasile Cîrtoaje*, GM-B, 11, 2002)

40. If a, b, c, d are positive numbers, then

$$5 + \sqrt{2(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 2} \geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

(*Vasile Cîrtoaje*, GM-B, 5, 2004)

41. If a, b, c, d are positive numbers, then

$$\frac{a - b}{b + c} + \frac{b - c}{c + d} + \frac{c - d}{d + a} + \frac{d - a}{a + b} \geq 0.$$

42. If $a, b, c > -1$, then

$$\frac{1 + a^2}{1 + b + c^2} + \frac{1 + b^2}{1 + c + a^2} + \frac{1 + c^2}{1 + a + b} \geq 2.$$

(*Laurenţiu Panaitopol*, Junior BMO, 2003)

43. Let a, b, c and x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

(*Vasile Cîrtoaje, Mircea Lascu*, ONI, 1996)

44. Let a, b, c be positive numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \ge 3.$$

(*Cezar Lupu*, MS, 2005)

45. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \ge \frac{3}{ab+bc+ca}.$$

(*Vasile Cîrtoaje*, MS, 2005)

46. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b^2-bc+c^2} + \frac{1}{c^2-ca+a^2} + \frac{1}{a^2-ab+b^2} \ge \frac{3}{ab+bc+ca}.$$

47. Let a, b, c be positive numbers such that $a+b+c = 3$. Prove that

$$abc + \frac{12}{ab+bc+ca} \ge 5.$$

48. Let a, b, c be non-negative numbers such that $a^2+b^2+c^2 = 3$. Prove that

$$12 + 9abc \ge 7(ab+bc+ca).$$

(*Vasile Cîrtoaje*, MS, 2005)

49. Let a, b, c be non-negative numbers such that $ab+bc+ca = 3$. Prove that

$$a^3+b^3+c^3+7abc \ge 10.$$

(*Vasile Cîrtoaje*, MS, 2005)

50. If a, b, c are positive numbers such that $abc = 1$, then

$$(a+b)(b+c)(c+a)+7 \geq 5(a+b+c).$$

(*Vasile Cîrtoaje*, MS, 2005)

51. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)}+\frac{b^3}{(2b^2+c^2)(2b^2+a^2)}+\frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \leq \frac{1}{a+b+c}.$$

(*Vasile Cîrtoaje*, MS, 2005)

52. Let a, b, c be non-negative numbers such that $a+b+c \geq 3$. Prove that

$$\frac{1}{a^2+b+c}+\frac{1}{a+b^2+c}+\frac{1}{a+b+c^2} \leq 1.$$

53. Let a, b, c be non-negative numbers such that $ab+bc+ca = 3$. If $r \geq 1$, then

$$\frac{1}{r+a^2+b^2}+\frac{1}{r+b^2+c^2}+\frac{1}{r+c^2+a^2} \leq \frac{3}{r+2}.$$

(*Pham Van Thuan*, MS, 2005)

54. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{(1+a)^3}+\frac{1}{(1+b)^3}+\frac{1}{(1+c)^3}+\frac{5}{(1+a)(1+b)(1+c)} \geq 1.$$

(*Pham Kim Hung*, MS, 2006)

55. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{2}{a+b+c}+\frac{1}{3} \geq \frac{3}{ab+bc+ca}.$$

56. If a, b, c are real numbers, then

$$2(1+abc)+\sqrt{2(1+a^2)(1+b^2)(1+c^2)} \geq (1+a)(1+b)(1+c).$$

(*Wolfgang Berndt*, MS, 2006)

57. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \geq 2.$$

(*Pham Kim Hung*, MS, 2006)

58. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}} \ge 2.$$

(*Vasile Cîrtoaje*, MS, 2006)

59. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \ge \frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}.$$

60. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \ge \frac{2a}{3a^2+bc}+\frac{2b}{3b^2+ca}+\frac{2c}{3c^2+ab}.$$

(*Vasile Cîrtoaje*, MS, 2005)

61. Let a, b, c be positive numbers such that $a^2+b^2+c^2=3$. Prove that

$$5(a+b+c)+\frac{3}{abc} \ge 18.$$

(*Vasile Cîrtoaje*, MS, 2005)

62. Let a, b, c be non-negative numbers such that $a+b+c=3$. Prove that

$$\frac{1}{6-ab}+\frac{1}{6-bc}+\frac{1}{6-ca} \le \frac{3}{5}.$$

63. Let $n \ge 4$ and let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1+a_2+\cdots+a_n \ge n \quad \text{and} \quad a_1^2+a_2^2+\cdots+a_n^2 \ge n^2.$$

Prove that

$$\max\{a_1, a_2, \ldots, a_n\} \ge 2.$$

(*Titu Andreescu*, USAMO, 1999)

64. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{13}{6}-\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}.$$

(*Vasile Cîrtoaje*, MS, 2006)

65. Let a, b, c be non-negative numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq a+b+c.$$

(*Darij Grinberg*, MS, 2004)

66. Let a, b, c be non-negative numbers such that

$$(a+b)(b+c)(c+a) = 2.$$

Prove that

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq 1.$$

(*Vasile Cîrtoaje*, MS, 2005)