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Chuc Mung Nam Moi 2008

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Inequalities From 2007 Mathematical Competition Over The World

★ ★ ★

Example 1 (Iran National Mathematical Olympiad 2007). *Assume that a, b, c are three different positive real numbers. Prove that*

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>1.$$

Example 2 (Iran National Mathematical Olympiad 2007). *Find the largest real T such that for each non-negative real numbers a, b, c, d, e such that $a+b=c+d+e$, then*

$$\sqrt{a^2+b^2+c^2+d^2+e^2} \geq T(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}+\sqrt{e})^2.$$

Example 3 (Middle European Mathematical Olympiad 2007). *Let a, b, c, d be positive real numbers with $a+b+c+d=4$. Prove that*

$$a^2bc+b^2cd+c^2da+d^2ab \leq 4.$$

Example 4 (Middle European Mathematical Olympiad 2007). *Let a, b, c, d be real numbers which satisfy $\frac{1}{2} \leq a, b, c, d \leq 2$ and $abcd=1$. Find the maximum value of*

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right).$$

Example 5 (China Northern Mathematical Olympiad 2007). *Let a, b, c be side lengths of a triangle and $a+b+c=3$. Find the minimum of*

$$a^2+b^2+c^2+\frac{4abc}{3}.$$

Example 6 (China Northern Mathematical Olympiad 2007). *Let α, β be acute angles. Find the maximum value of*

$$\frac{\left(1-\sqrt{\tan\alpha\tan\beta}\right)^2}{\cot\alpha+\cot\beta}.$$

Example 7 (China Northern Mathematical Olympiad 2007). *Let a, b, c be positive real numbers such that $abc=1$. Prove that*

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a} \geq \frac{3}{2},$$

for any positive integer $k \geq 2$.

Example 8 (Croatia Team Selection Test 2007). *Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3(a^2 + b^2 + c^2).$$

Example 9 (Romania Junior Balkan Team Selection Tests 2007). *Let a, b, c three positive reals such that*

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Show that

$$a + b + c \geq ab + bc + ca.$$

Example 10 (Romania Junior Balkan Team Selection Tests 2007). *Let $x, y, z \geq 0$ be real numbers. Prove that*

$$\frac{x^3 + y^3 + z^3}{3} \geq xyz + \frac{3}{4}|(x-y)(y-z)(z-x)|.$$

Example 11 (Yugoslavia National Olympiad 2007). *Let k be a given natural number. Prove that for any positive numbers x, y, z with the sum 1 the following inequality holds*

$$\frac{x^{k+2}}{x^{k+1} + y^k + z^k} + \frac{y^{k+2}}{y^{k+1} + z^k + x^k} + \frac{z^{k+2}}{z^{k+1} + x^k + y^k} \geq \frac{1}{7}.$$

Example 12 (Cezar Lupu & Tudorel Lupu, Romania TST 2007). *For $n \in \mathbb{N}, n \geq 2, a_i, b_i \in \mathbb{R}, 1 \leq i \leq n$, such that $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2 = 1, \sum_{i=1}^{n} a_i b_i = 0$. Prove that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$$

Example 13 (Macedonia Team Selection Test 2007). *Let a, b, c be positive real numbers. Prove that*

$$1 + \frac{3}{ab + bc + ca} \geq \frac{6}{a + b + c}.$$

Example 14 (Italian National Olympiad 2007). *a) For each $n \geq 2$, find the maximum constant c_n such that*

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \ldots + \frac{1}{a_n + 1} \geq c_n,$$

for all positive reals $a_1, a_2, \ldots, a_n$ such that $a_1 a_2 \cdots a_n = 1$.

b) For each $n \geq 2$, find the maximum constant d_n such that

$$\frac{1}{2a_1 + 1} + \frac{1}{2a_2 + 1} + \ldots + \frac{1}{2a_n + 1} \geq d_n$$

for all positive reals $a_1, a_2, \ldots, a_n$ such that $a_1 a_2 \cdots a_n = 1$.

Example 15 (France Team Selection Test 2007). *Let a, b, c, d be positive reals such taht $a+b+c+d=1$. Prove that*

$$6(a^3+b^3+c^3+d^3) \geq a^2+b^2+c^2+d^2+\frac{1}{8}.$$

Example 16 (Irish National Mathematical Olympiad 2007). *Suppose a, b and c are positive real numbers. Prove that*

$$\frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} \leq \frac{1}{3}\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right).$$

For each of the inequalities, find conditions on a, b and c such that equality holds.

Example 17 (Vietnam Team Selection Test 2007). *Given a triangle ABC. Find the minimum of*

$$\frac{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}+\frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{\cos^2\frac{C}{2}\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}.$$

Example 18 (Greece National Olympiad 2007). *Let a,b,c be sides of a triangle, show that*

$$\frac{(c+a-b)^4}{a(a+b-c)}+\frac{(a+b-c)^4}{b(b+c-a)}+\frac{(b+c-a)^4}{c(c+a-b)} \geq ab+bc+ca.$$

Example 19 (Bulgaria Team Selection Tests 2007). *Let $n \geq 2$ is positive integer. Find the best constant $C(n)$ such that*

$$\sum_{i=1}^{n} x_i \geq C(n) \sum_{1\leq j<i\leq n} (2x_ix_j+\sqrt{x_ix_j})$$

is true for all real numbers $x_i \in (0,1), i=1,...,n$ for which $(1-x_i)(1-x_j) \geq \frac{1}{4}, 1 \leq j < i \leq n$.

Example 20 (Poland Second Round 2007). *Let a, b, c, d be positive real numbers satisfying the following condition:*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=4.$$

Prove that:

$$\sqrt[3]{\frac{a^3+b^3}{2}}+\sqrt[3]{\frac{b^3+c^3}{2}}+\sqrt[3]{\frac{c^3+d^3}{2}}+\sqrt[3]{\frac{d^3+a^3}{2}} \leq 2(a+b+c+d)-4.$$

Example 21 (Turkey Team Selection Tests 2007). *Let a, b, c be positive reals such that their sum is 1. Prove that*

$$\frac{1}{ab+2c^2+2c}+\frac{1}{bc+2a^2+2a}+\frac{1}{ac+2b^2+2b} \geq \frac{1}{ab+bc+ac}.$$

Example 22 (Moldova National Mathematical Olympiad 2007). *Real numbers* $a_1, a_2, \ldots, a_n$ *satisfy* $a_i \geq \dfrac{1}{i}$, *for all* $i = \overline{1, n}$. *Prove the inequality*

$$(a_1+1)\left(a_2+\frac{1}{2}\right)\cdot\cdots\cdot\left(a_n+\frac{1}{n}\right) \geq \frac{2^n}{(n+1)!}(1+a_1+2a_2+\cdots+na_n).$$

Example 23 (Moldova Team Selection Test 2007). *Let* $a_1, a_2, \ldots, a_n \in [0,1]$. *Denote* $S = a_1^3 + a_2^3 + \ldots + a_n^3$, *prove that*

$$\frac{a_1}{2n+1+S-a_1^3} + \frac{a_2}{2n+1+S-a_2^3} + \ldots + \frac{a_n}{2n+1+S-a_n^3} \leq \frac{1}{3}.$$

Example 24 (Peru Team Selection Test 2007). *Let* a, b, c *be positive real numbers, such that*

$$a+b+c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove that

$$a+b+c \geq \frac{3}{a+b+c} + \frac{2}{abc}.$$

Example 25 (Peru Team Selection Test 2007). *Let* a, b *and* c *be sides of a triangle. Prove that*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.'$$

Example 26 (Romania Team Selection Tests 2007). *If* $a_1, a_2, \ldots, a_n \geq 0$ *satisfy* $a_1^2 + \cdots + a_n^2 = 1$, *find the maximum value of the product* $(1-a_1)\cdots(1-a_n)$.

Example 27 (Romania Team Selection Tests 2007). *Prove that for* n, p *integers,* $n \geq 4$ *and* $p \geq 4$, *the proposition* $\mathcal{P}(n,p)$

$$\sum_{i=1}^{n} \frac{1}{{x_i}^p} \geq \sum_{i=1}^{n} {x_i}^p \quad for \quad x_i \in \mathbb{R}, \quad x_i > 0, \quad i = 1, \ldots, n\ , \quad \sum_{i=1}^{n} x_i = n,$$

is false.

Example 28 (Ukraine Mathematical Festival 2007). *Let* a, b, c *be positive real numbers and* $abc \geq 1$. *Prove that*

(a).

$$\left(a+\frac{1}{a+1}\right)\left(b+\frac{1}{b+1}\right)\left(c+\frac{1}{c+1}\right) \geq \frac{27}{8}.$$

(b).

$$27(a^3+a^2+a+1)(b^3+b^2+b+1)(c^3+c^2+c+1) \geq\geq 64(a^2+a+1)(b^2+b+1)(c^2+c+1).$$

Example 29 (Asian Pacific Mathematical Olympiad 2007). *Let* x, y *and* z *be positive real numbers such that* $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. *Prove that*

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}} + \frac{y^2+zx}{\sqrt{2y^2(z+x)}} + \frac{z^2+xy}{\sqrt{2z^2(x+y)}} \geq 1.$$

Example 30 (Brazilian Olympiad Revenge 2007). *Let* $a, b, c \in \mathbb{R}$ *with* $abc = 1$. *Prove that*

$$a^2+b^2+c^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 6+2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+\frac{c}{a}+\frac{c}{b}+\frac{b}{c}\right).$$

Example 31 (India National Mathematical Olympiad 2007). *If* x, y, z *are positive real numbers, prove that*

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

Example 32 (British National Mathematical Olympiad 2007). *Show that for all positive reals* a, b, c,

$$(a^2+b^2)^2 \geq (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

Example 33 (Korean National Mathematical Olympiad 2007). *For all positive reals* $a, b,$ *and* c, *what is the value of positive constant k satisfies the following inequality?*

$$\frac{a}{c+kb}+\frac{b}{a+kc}+\frac{c}{b+ka} \geq \frac{1}{2007}.$$

Example 34 (Hungary-Isarel National Mathematical Olympiad 2007). *Let* a, b, c, d *be real numbers, such that*

$$a^2 \leq 1, a^2+b^2 \leq 5, a^2+b^2+c^2 \leq 14, a^2+b^2+c^2+d^2 \leq 30.$$

Prove that $a+b+c+d \leq 10$.

SOLUTION

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Please visit the following links to get the original discussion of the ebook, the problems and solution. We are appreciating every other contribution from you!

http://www.batdangthuc.net/forum/showthread.php?t=26

http://www.batdangthuc.net/forum/showthread.php?t=26&page=2

http://www.batdangthuc.net/forum/showthread.php?t=26&page=3

http://www.batdangthuc.net/forum/showthread.php?t=26&page=4

http://www.batdangthuc.net/forum/showthread.php?t=26&page=5

http://www.batdangthuc.net/forum/showthread.php?t=26&page=6

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For Further Reading, Please Review:

★ UpComing Vietnam Inequality Forum's Magazine

★ Secrets in Inequalities (2 volumes), Pham Kim Hung (hungkhtn)

★ Old And New Inequalities, T. Adreescu, V. Cirtoaje, M. Lascu, G. Dospinescu

★ Inequalities and Related Issues, Nguyen Van Mau

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We thank a lot to Mathlinks Forum and their member for the reference to problems and some nice solutions from them!

Problem 1 (1, Iran National Mathematical Olympiad 2007). *Assume that a, b, c are three different positive real numbers. Prove that*

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>1.$$

Solution 1 (pi3.14). Due to the symmetry, we can assume $a>b>c$. Let $a=c+x; b=c+y$, then $x>y>0$. We have

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|$$

$$=\frac{2c+x+y}{x-y}+\frac{2c+y}{y}-\frac{2c+x}{x}$$

$$=2c\left(\frac{1}{x-y}+\frac{1}{y}-\frac{1}{x}\right)+\frac{x+y}{x-y}.$$

We have

$$2c\left(\frac{1}{x-y}+\frac{1}{y}-\frac{1}{x}\right)=2c\left(\frac{1}{x-y}+\frac{x-y}{xy}\right)>0.$$

$$\frac{x+y}{x-y}>1.$$

Thus

$$\left|\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}\right|>1.$$

Solution 2 (2, Mathlinks, posted by NguyenDungTN). Let

$$\frac{a+b}{a-b}=x; \frac{b+c}{b-c}=y; \frac{a+c}{c-a}=z;$$

Then

$$xy+yz+xz=1.$$

By Cauchy-Schwarz Inequality

$$(x+y+z)^2\geq 3(xy+yz+zx)=3\Rightarrow |x+y+z|\geq\sqrt{3}>1.$$

We are done.

∇

Problem 2 (2, Iran National Mathematical Olympiad 2007). *Find the largest real T such that for each non-negative real numbers a, b, c, d, e such that $a+b=c+d+e$, then*

$$\sqrt{a^2+b^2+c^2+d^2+e^2}\geq T(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}+\sqrt{e})^2$$

Solution 3 (NguyenDungTN). Let $a = b = 3, c = d = e = 2$, we find

$$\frac{\sqrt{30}}{6(\sqrt{3}+\sqrt{2})^2} \geq T.$$

With this value of T, we will prove the inequality. Indeed, let $a + b = c + d + e = X$. By Cauchy-Schwarz Inequality

$$a^2+b^2 \geq \frac{(a+b)^2}{2} = \frac{X^2}{2} c^2+d^2+e^2 \geq \frac{(c+d+e)^2}{3} = \frac{X^2}{3}$$

$$\Rightarrow \sqrt{a^2+b^2+c^2+d^2+e^2} \geq \frac{5X^2}{6} \quad (1)$$

By Cauchy-Schwarz Inequality, we also have

$$\sqrt{a}+\sqrt{b} \leq \sqrt{2(a+b)} = \sqrt{2X}\sqrt{c}+\sqrt{d}+\sqrt{e} \leq \sqrt{3(c+d+e)=3X}$$

$$\Rightarrow (\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}+\sqrt{e})^2 \leq (\sqrt{2}+\sqrt{3})^2 X^2 \quad (2)$$

From (1) and (2), we have

$$\frac{\sqrt{a^2+b^2+c^2+d^2+e^2}}{(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}+\sqrt{e})^2} \geq \frac{\sqrt{30}}{6(\sqrt{3}+\sqrt{2})^2}.$$

Equality holds for $\frac{2a}{3} = \frac{2b}{3} = c = d = e.$

∇

Problem 3 (3, Middle European Mathematical Olympiad 2007). *Let a, b, c, d non-negative such that $a + b + c + d = 4$. Prove that*

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4.$$

Solution 4 (mathlinks, reposted by pi3.14). Let $\{p, q, r, s\} = \{a, b, c, d\}$ and $p \geq q \geq r \geq s$. By rearrangement Inequality, we have

$$a^2bc + b^2cd + c^2da + d^2ab = a(abc) + b(bcd) + c(cda) + d(dab)$$

$$\leq p(pqr) + q(pqs) + r(prs) + s(qrs) = (pq + rs)(pr + qs)$$

$$\leq \left(\frac{pq+rs+pr+qs}{2}\right)^2 = \frac{1}{4}(p+s)^2(q+r)^2$$

$$\leq \frac{1}{4}\left(\left(\frac{p+q+r+s}{2}\right)^2\right)^2 = 4.$$

Equality holds for $q = r = 1vp + s = 2$. Easy to refer $(a, b, c, d) = (1, 1, 1, 1), (2, 1, 1, 0)$ or permutations.

∇

Problem 4 (5- Revised by VanDHKH). *Let a, b, c be three side-lengths of a triangle such that $a+b+c=3$. Find the minimum of $a^2+b^2+c^2+\dfrac{4abc}{3}$*

Solution 5. Let $a=x+y, b=y+z, c=z+x$, we have

$$x+y+z=\frac{3}{2}.$$

Consider

$$\begin{aligned}
&a^2+b^2+c^2+\frac{4abc}{3}\\
&=\frac{(a^2+b^2+c^2)(a+b+c)+4abc}{3}\\
&=\frac{2((x+y)^2+(y+z)^2+(z+x)^2)(x+y+z)+4(x+y)(y+z)(z+x)}{3}\\
&=\frac{4(x^3+y^3+z^3+3x^2y+3xy^2+3y^2z+3yz^2+3z^2x+3zx^2+5xyz)}{3}\\
&=\frac{4((x+y+z)^3-xyz)}{3}\\
&=\frac{4(\frac{26}{27}(x+y+z)^3+(\frac{x+y+z}{3})^3-xyz)}{3}\\
&\geq\frac{4(\frac{26}{27}(x+y+z)^3)}{3}=\frac{13}{3}.
\end{aligned}$$

Solution 6 (2, DDucLam). Using the familiar Inequality (equivalent to Schur)

$$abc\geq(b+c-a)(c+a-b)(a+b-c)\ \Rightarrow abc\geq\frac{4}{3}(ab+bc+ca)-3.$$

Therefore

$$P\geq a^2+b^2+c^2+\frac{16}{9}(ab+bc+ca)-4$$

$$=(a+b+c)^2-\frac{2}{9}(ab+bc+ca)-4\geq 5-\frac{2}{27}(a+b+c)^2=4+\frac{1}{3}.$$

Equality holds when $a=b=c=1$.

Solution 7 (3, pi3.14). With the conventional denotion in triangle, we have

$$abc=4pRr\ ,\ a^2+b^2+c^2=2p^2-8Rr-2r^2.$$

Therefore

$$a^2+b^2+c^2+\frac{4}{3}abc=\frac{9}{2}-2r^2.$$

Moreover,

$$p\geq 3\sqrt{3}r\ \Rightarrow r^2\leq\frac{1}{6}.$$

Thus

$$a^2+b^2+c^2+\frac{4}{3}abc\geq 4\frac{1}{3}.$$

∇

Problem 5 (7, China Northern Mathematical Olympiad 2007). *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a}\ge\frac{3}{2}.$$

for any positive integer $k \ge 2$.

Solution 8 (Secrets In Inequalities, hungkhtn). We have

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a}\ge\frac{3}{2}$$

$$\Leftrightarrow a^{k-1}+b^{k-1}+c^{k-1}\ge\frac{3}{2}+\frac{a^{k-1}b}{a+b}+\frac{b^{k-1}c}{b+c}+\frac{c^{k-1}a}{c+a}$$

By AM-GM Inequality, we have

$$a+b\ge 2\sqrt{ab}, b+c\ge 2\sqrt{bc}, c+a\ge 2\sqrt{ca}.$$

So, it remains to prove that

$$a^{k-\frac{3}{2}}b^{\frac{1}{2}}+b^{k-\frac{3}{2}}c^{\frac{1}{2}}+c^{k-\frac{3}{2}}a^{\frac{1}{2}}+3\le 2\left(a^{k-1}+b^{k-1}+c^{k-1}\right).$$

This follows directly by AM-GM inequality, since

$$a^{k-1}+b^{k-1}+c^{k-1}\ge 3\sqrt[3]{a^{k-1}b^{k-1}c^{k-1}}=3$$

and

$$(2k-3)a^{k-1}+b^{k-1}\ge(2k-2)a^{k-\frac{3}{2}}b^{\frac{1}{2}}$$

$$(2k-3)b^{k-1}+c^{k-1}\ge(2k-2)b^{k-\frac{3}{2}}c^{\frac{1}{2}}$$

$$(2k-3)c^{k-1}+a^{k-1}\ge(2k-2)c^{k-\frac{3}{2}}a^{\frac{1}{2}}$$

Adding up these inequalities, we have the desired result.

∇

Problem 6 (8, Revised by NguyenDungTN). *Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that:*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3(a^2+b^2+c^2).$$

Solution 9. By Cauchy-Schwarz Inequality:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}.$$

It remains to prove that

$$\begin{aligned} &\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a} \geq 3(a^2+b^2+c^2) \\ &\Leftrightarrow (a^2+b^2+c^2)(a+b+c) \geq 3(a^2b+b^2c+c^2a) \\ &\Leftrightarrow a^3+b^3+c^3+ab^2+bc^2+ca^2 \geq 2(a^2b+b^2c+c^2a) \\ &\Leftrightarrow a(a-b)^2+b(b-c)^2+c(c-a)^2 \geq 0. \end{aligned}$$

So we are done!

Solution 10 (2, By Zaizai).

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3(a^2+b^2+c^2)$$

$$\Leftrightarrow \sum \left(\frac{a^2}{b} - 2a + b\right) \geq 3(a^2+b^2+c^2) - (a+b+c)^2$$

$$\Leftrightarrow \sum \frac{(a-b)^2}{b} \geq (a-b)^2 + (b-c)^2 + (c-a)^2$$

$$\Leftrightarrow \sum (a-b)^2 \left(\frac{1}{b} - 1\right) \geq 0$$

$$\Leftrightarrow \sum \frac{(a-b)^2(a+c)}{b} \geq 0.$$

This ends the solution, too.

∇

Problem 7 (9, Romania Junior Balkan Team Selection Tests 2007). *. Let a, b, c be three positive reals such that*

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Show that

$$a+b+c \geq ab+bc+ca.$$

Solution 11 (Mathlinks, Reposted by NguyenDungTN). By Cauchy-Schwarz Inequality, we have

$$(a+b+1)(a+b+c^2) \geq (a+b+c)^2.$$

Therefore
$$\frac{1}{a+b+1} \le \frac{c^2+a+b}{(a+b+c)^2},$$
or
$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \le \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2}$$
$$\Rightarrow a^2+b^2+c^2+2(a+b+c) \ge (a+b+c)^2$$
$$\Rightarrow a+b+c \ge ab+bc+ca.$$

Solution 12 (DDucLam). Assume that $a+b+c = ab+bc+ca$, we have to prove that
$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \le 1$$
$$\Leftrightarrow \frac{a+b}{a+b+1}+\frac{b+c}{b+c+1}+\frac{c+a}{c+a+1} \ge 2$$
By Cauchy-Schwarz Inequality,
$$\text{LHS} \ge \frac{(a+b+b+c+c+a)^2}{\sum_{cyc}(a+b)(a+b+1)} = 2.$$
We are done

Comment. This second very beautiful solution uses Contradiction method. If you can't understand the principal of this method, have a look at *Sang Tao Bat Dang Thuc*, or *Secrets In Inequalities*, written by Pham Kim Hung.

∇

Problem 8 (10, Romanian JBTST V 2007). *Let x, y, z be non-negative real numbers. Prove that*
$$\frac{x^3+y^3+z^3}{3} \ge xyz + \frac{3}{4}|(x-y)(y-z)(z-x)|.$$
Solution 13 (vandhkh). We have
$$\frac{x^3+y^3+z^3}{3} \ge xyz + \frac{3}{4}|(x-y)(y-z)(z-x)|$$
$$\Leftrightarrow \frac{x^3+y^3+z^3}{3} - xyz \ge \frac{3}{4}|(x-y)(y-z)(z-x)|$$
$$\Leftrightarrow ((x-y)^2+(y-z)^2+(z-x)^2(((x+y)+(y+z)+(z+x)) \ge 9|(x-y)(y-z)(z-x)|.$$
Notice that
$$x+y \ge |x-y|; y+z \ge |y-z|; z+x \ge |z-x|,$$
and by AM-GM Inequality,
$$((x-y)^2+(y-z)^2+(z-x)^2)(|x-y|+|y-z|+|z-x|) \ge 9|(x-y)(y-z)(z-x)|.$$
So we are done. Equality holds for $x=y=z$.

Solution 14 (Secrets In Inequalities, hungkhtn). The inequality is equivalent to

$$(x+y+z)\sum(x-y)^2 \geq \frac{9}{2}|(x-y)(y-z)(z-x)|.$$

By the entirely mixing variable method, it is enough to prove when $z=0$

$$x^3+y^3 \geq \frac{9}{4}|xy(x-y)|.$$

This last inequality can be checked easily.

∇

Problem 9 (11, Yugoslavia National Olympiad 2007). *Let k be a given natural number. Prove that for any positive numbers x, y, z with the sum 1, the following inequality holds*

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k} \geq \frac{1}{7}.$$

When does equality occur?

Solution 15 (NguyenDungTN). We can assume that $x \geq y \geq z$. By this assumption, easy to refer that

$$\frac{x^{k+1}}{x^{k+1}+y^k+z^k} \geq \frac{y^{k+1}}{y^{k+1}+z^k+x^k} \geq \frac{z^{k+1}}{z^{k+1}+x^k+y^k}\ ;$$

$$z^{k+1}+y^k+x^k \geq y^{k+1}+x^k+z^k \geq x^{k+1}+z^k+y^k\ ;$$

and

$$x^k \geq y^k \geq z^k.$$

By Chebyshev Inequality, we have

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k}$$

$$\geq \frac{x+y+z}{3}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}+\frac{y^{k+1}}{y^{k+1}+z^k+x^k}+\frac{z^{k+1}}{z^{k+1}+x^k+y^k}\right)$$

$$=\frac{1}{3}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}+\frac{y^{k+1}}{y^{k+1}+z^k+x^k}+\frac{z^{k+1}}{z^{k+1}+x^k+y^k}\right)\frac{\sum_{cyc}(x^{k+1}+y^k+z^k)}{\sum_{cyc}(x^{k+1}+y^k+z^k)}$$

$$=\frac{1}{3}\left(\sum_{cyc}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}\sum_{cyc}(x^{k+1}+y^k+z^k)\frac{1}{\sum_{cyc}(x^{k+1}+y^k+z^k)}\right)\right)$$

$$\geq \frac{1}{3}(x^{k+1}+y^{k+1}+z^{k+1}).\frac{1}{\sum_{cyc}(x^{k+1}+y^k+z^k)}=\frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2(x^k+y^k+z^k)}$$

Also by Chebyshev Inequality,

$$3(x^{k+1}+y^{k+1}+z^{k+1}) \geq 3\frac{x+y+z}{3}(x^k+y^k+z^k) = x^k+y^k+z^k.$$

Thus

$$\frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2(x^k+y^k+z^k)} \geq \frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+6(x^{k+1}+y^{k+1}+z^{k+1})} = \frac{1}{7}.$$

So we are done. Equality holds for $a=b=c=\frac{1}{3}$.

∇

Problem 10 (Macedonia Team Selection Test 2007). *Let a, b, c be positive real numbers. Prove that*

$$1+\frac{3}{ab+bc+ca} \geq \frac{6}{a+b+c}.$$

Solution 16 (VoDanh). The inequality is equivalent to

$$a+b+c+\frac{3(a+b+c)}{ab+bc+ca} \geq 6.$$

By AM-GM Inequality,

$$a+b+c+\frac{3(a+b+c)}{ab+bc+ca} \geq 2\sqrt{\frac{3(a+b+c)^2}{ab+bc+ca}}.$$

It is obvious that $(a+b+c)^2 \geq 3(ab+bc+ca)$, so we are done!

∇

Problem 11 (14, Italian National Olympiad 2007). *a). For each $n \geq 2$, find the maximum constant c_n such that:*

$$\frac{1}{a_1+1}+\frac{1}{a_2+1}+\ldots+\frac{1}{a_n+1} \geq c_n,$$

for all positive reals $a_1, a_2, \ldots, a_n$ such that $a_1a_2\cdots a_n = 1$.

∇

b). For each $n \geq 2$, find the maximum constant d_n such that

$$\frac{1}{2a_1+1}+\frac{1}{2a_2+1}+\ldots+\frac{1}{2a_n+1} \geq d_n,$$

for all positive reals $a_1, a_2, \ldots, a_n$ such that $a_1a_2\cdots a_n = 1$.

Solution 17 (Mathlinks, reposted by NguyenDungTN). a). Let

$$a_1 = \epsilon^{n-1}, a_k = \frac{1}{\epsilon} \forall k \neq 1,$$

then let $\epsilon \to 0$, we easily get $c_n \leq 1$. We will prove the inequality with this value of c_n. Without loss of generality, assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since $a_1 a_2 \leq 1$, we have

$$\sum_{k=1}^{n} \frac{1}{a_k+1} \geq \frac{1}{a_1+1} + \frac{1}{a_2+1} = \frac{1}{a_1+1} + \frac{a_1}{a_2+a_1a_2} \geq \frac{1}{a_1+1} + \frac{a_1}{a_1+1} = 1.$$

This ends the proof.

b). Consider $n = 2$, it is easy to get $d_2 = \frac{2}{3}$. Indeed, let $a_1 = a, a_2 = \frac{1}{a}$. The inequality becomes

$$\frac{1}{2a+1} + \frac{a}{a+2} \geq \frac{2}{3} \Leftrightarrow 3(a+2) + 3a(2a+1) \geq 2(2a+1)(a+2)$$

$$\Leftrightarrow (a-1)^2 \geq 0.$$

When $n \geq 3$, similar to (a), we will show that $d_n = 1$. Indeed, without loss of generality, we may assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n \ \Rightarrow a_1a_2a_3 \leq 1.$$

Let

$$x = \sqrt[9]{\frac{a_2a_3}{a_1^2}},\ y = \sqrt[9]{\frac{a_1a_3}{a_2^2}},\ z = \sqrt[9]{\frac{a_1a_2}{a_3^2}}$$

then $a_1 \leq \frac{1}{x^3},\ a_2 \leq \frac{1}{y^3},\ a_3 \leq \frac{1}{z^3}, xyz = 1$. Thus

$$\sum_{k=1}^{n} \frac{1}{a_k+1} \geq \sum_{k=1}^{3} \frac{1}{a_k+1} \geq \frac{x^3}{x^3+2} + \frac{y^3}{y^3+2} + \frac{z^3}{z^3+2}$$

$$= \frac{x^2}{x^2+2yz} + \frac{y^2}{y^2+2xz} + \frac{z^2}{z^2+2xy}$$

$$\geq \frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2} = 1.$$

This ends the proof.

∇

Problem 12 (15, France Team Selection Test 2007). *. Let a, b, c, d be positive reals such that $a+b+c+d = 1$. Prove that:*

$$6(a^3+b^3+c^3+d^3) \geq a^2+b^2+c^2+d^2+\frac{1}{8}.$$

Solution 18 (NguyenDungTN). By AM-GM Inequality

$$2a^3+\frac{1}{4^3}\geq\frac{3a^2}{4}a^2+\frac{1}{4^2}\geq\frac{a}{2}.$$

Therefore

$$6(a^3+b^3+c^3+d^3)+\frac{3}{16}\geq\frac{9(a^2+b^2+c^2+d^2)}{4}$$
$$\frac{5(a^2+b^2+c^2+d^2)}{4}+\frac{5}{16}\geq\frac{5(a+b+c+d}{8}=\frac{5}{8}$$

Adding up two of them, we get

$$6(a^3+b^3+c^3+d^3)\geq a^2+b^2+c^2+d^2+\frac{1}{8}$$

Solution 19 (Zaizai). We known that

$$6a^3\geq a^2+\frac{5a}{8}-\frac{1}{8}\Leftrightarrow\frac{(4a-1)^2(3a+1)}{8}\geq 0$$

Adding up four similar inequalities, we are done!

∇

Problem 13 (16, Revised by NguyenDungTN). *Suppose a, b and c are positive real numbers. Prove that*

$$\frac{a+b+c}{3}\leq\sqrt{\frac{a^2+b^2+c^2}{3}}\leq\frac{1}{3}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right).$$

Solution 20. The left-hand inequality is just Cauchy-Schwarz Inequality. We will prove the right one. Let

$$\frac{bc}{a}=x,\frac{ca}{b}=y,\frac{ab}{c}=z.$$

The inequality becomes

$$\sqrt{\frac{xy+yz+zx}{3}}\leq\frac{x+y+z}{3}.$$

Squaring both sides, the inequality becomes

$$(x+y+z)^2\geq 3(xy+yz+zx)\Leftrightarrow(x-y)^2+(y-z)^2+(z-x)^2\geq 0,$$

which is obviously true.

∇

Problem 14 (17, Vietnam Team Selection Test 2007). *Given a triangle ABC. Find the minimum of:*

$$\sum\frac{(cos^2(\frac{A}{2})(cos^2(\frac{B}{2})}{cos^2(\frac{C}{2})}$$

Solution 21 (pi3.14). We have

$$T=\sum\frac{(cos^2(\frac{A}{2})(cos^2(\frac{B}{2})}{(cos^2(\frac{C}{2})}$$

$$=\sum\frac{(1+cosA)(1+cosB)}{2(1+cosC)}.$$

Let $a=tan\frac{A}{2};b=tan\frac{B}{2};c=tan\frac{C}{2}$. We have $ab+bc+ca=1$. So

$$T=\sum\frac{(1+a^2)}{(1+b^2)(1+c^2)}=\sum\frac{1}{\frac{(1+b^2)(1+c^2)}{1+a^2}}$$

$$=\sum\frac{1}{\frac{(ab+bc+ca+b^2)(ab+bc+ca+c^2)}{(ab+bc+ca+a^2)}}$$

$$=\sum\frac{1}{\frac{(a+b)(c+b)(a+c)(b+c)}{(b+a)(b+c)}}$$

$$=\sum\frac{1}{(b+c)^2}$$

By Iran96 Inequality, we have

$$\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\geq\frac{9}{4(ab+bc+ca)}.$$

Thus $F\geq\frac{9}{4}$ Equality holds when ABC is equilateral.

∇

Problem 15 (18, Greece National Olympiad 2007). . *Let a,b,c be sides of a triangle, show that*

$$\frac{(b+c-a)^4}{a(a+b-c)}+\frac{(c+a-b)^4}{b(b+c-a)}+\frac{(b+c-a)^4}{a(c+a-b)}\geq ab+bc+ca.$$

Solution 22 (NguyenDungTN). Since a,b,c are three sides of a triangle, we can substitute

$$a=y+z,b=z+x,c=x+y.$$

The inequality becomes

$$\frac{8x^4}{(x+y)y}+\frac{8y^4}{(y+z)z}+\frac{8z^4}{(z+x)x}\geq x^2+y^2+z^2+3(xy+yz+zx).$$

By Cauchy-Schwarz Inequality, we have

$$\frac{8x^4}{(x+y)y}+\frac{8y^4}{(y+z)z}+\frac{8z^4}{(z+x)x}\geq\frac{8(x^2+y^2+z^2)^2}{x^2+y^2+z^2+xy+yz+zx}.$$

We will prove that

$$\frac{8(x^2+y^2+z^2)^2}{x^2+y^2+z^2+xy+yz+zx} \ge x^2+y^2+z^2+3(xy+yz+zx)$$

$$\Leftrightarrow 8(x^2+y^2+z^2)^2 \ge (x^2+y^2+z^2+xy+yz+zx)(x^2+y^2+z^2+3(xy+yz+zx))$$

$$\Leftrightarrow 8\sum x^4 + 16\sum x^2y^2 \ge \sum x^4 + 2\sum x^2y^2 +$$

$$+4\sum x^3(y+z) + 12xyz(x+y+z) + 3\sum x^2y^2 + 6xyz(x+y+z)$$

$$\Leftrightarrow 7\sum x^4 + 11\sum x^2y^2 \ge 4\sum x^3(y+z) + 10xyz(x+y+z).$$

By AM-GM and Schur Inequality

$$3\sum x^4 + 11\sum x^2y^2 \ge 14xyz(x+y+z);$$

$$4\left(\sum x^4 + xyz(x+y+z)\right) \ge 4\sum x^3(y+z)$$

Adding up two inequalities, we are done!

Solution 23 (2, DDucLam). By AM-GM Inequality, we have

$$\frac{(b+c-a)^4}{a(a+b-c)} + a(a+b-c) \ge 2(b+c-a)^2.$$

Construct two similar inequalities, then adding up, we have

$$\frac{(b+c-a)^4}{a(a+b-c)} + \frac{(c+a-b)^4}{b(b+c-a)} + \frac{(b+c-a)^4}{a(c+a-b)}$$

$$\ge 2[3(a^2+b^2+c^2) - 2(ab+bc+ca)] - (a^2+b^2+c^2)$$

$$= 5(a^2+b^2+c^2) - 4(ab+bc+ca) \ge ab+bc+ca.$$

We are done!

∇

Problem 16 (20, Poland Second Round 2007). . *Let a, b, c, d be positive real numbers satisfying the following condition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=4$ Prove that:*

$$\sqrt[3]{\frac{a^3+b^3}{2}} + \sqrt[3]{\frac{b^3+c^3}{2}} + \sqrt[3]{\frac{c^3+d^3}{2}} + \sqrt[3]{\frac{d^3+a^3}{2}} \le 2(a+b+c+d) - 4.$$

Solution 24 (Mathlinks, reposted by NguyenDungTN). First, we show that

$$\sqrt[3]{\frac{a^3+b^3}{2}} \le \frac{a^2+b^2}{a+b},$$

which is equivalent to

$$(a-b)^4(a^2+ab+b^2) \ge 0.$$

Therefore, we refer that

$$\sqrt[3]{\frac{a^3+b^3}{2}}+\sqrt[3]{\frac{b^3+c^3}{2}}+\sqrt[3]{\frac{c^3+d^3}{2}}+\sqrt[3]{\frac{d^3+a^3}{2}} \le \frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+d^2}{c+d}+\frac{d^2+a^2}{d+a}$$

It remains to prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+d^2}{c+d}+\frac{d^2+a^2}{d+a} \le 2(a+b+c+d)-4.$$

However,

$$a+b-\frac{a^2+b^2}{a+b} = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}},$$

So, due to Cauchy-Schwarz Inequality, we get

$$2(a+b+c+d)-\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+d^2}{c+d}+\frac{d^2+a^2}{d+a}\right)$$

$$= 2\sum \frac{1}{\frac{1}{a}+\frac{1}{b}} \ge 2\frac{4^2}{2(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})} = \frac{32}{8} = 4$$

This ends the proof.

∇

Problem 17 (21, Turkey Team Selection Tests 2007). . *Let a, b, c be positive reals such that their sum is 1. Prove that:*

$$\frac{1}{ab+2c^2+2c}+\frac{1}{bc+2a^2+2a}+\frac{1}{ac+2b^2+2b} \ge \frac{1}{ab+bc+ac}.$$

Solution 25 (NguyenDungTN). First, we will prove that

$$\frac{ab+ac+bc}{ab+2c^2+2c} \ge \frac{ab}{ab+ac+bc}.$$

Indeed, this is equivalent to

$$a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c) \ge a^2b^2+2abc^2+2abc,$$

which is always true since $2abc(a+b+c) = 2abc$ and due to AM-GM Inequality

$$a^2c^2+b^2c^2 \ge 2abc^2.$$

Similarly, we have

$$\frac{ab+ac+bc}{bc+2a^2+2a} \geq \frac{bc}{ab+ac+bc}.$$
$$\frac{ab+ac+bc}{ac+2b^2+2b} \geq \frac{ca}{ab+ac+bc}.$$

Adding up three inequalities, we are done!

∇

Problem 18 (22, Moldova National Mathematical Olympiad 2007). *Real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_i \geq \frac{1}{i}$, for all $i = \overline{1,n}$. Prove the inequality*

$$(a_1+1)\left(a_2+\frac{1}{2}\right)\cdot \cdots \cdot \left(a_n+\frac{1}{n}\right) \geq \frac{2^n}{(n+1)!}(1+a_1+2a_2+\cdots+na_n).$$

Solution 26 (NguyenDungTN). This inequality is equivalent to

$$(a_1+1)(2a_2+1)\cdot ..\cdot(na_n+1) \geq \frac{2^n}{n+1}(1+a_1+2a_2+...+na_n).$$

It is clearly true when $n=1$. Assume that it si true for $n=k$, we have to prove it for $n=k+1$. Indeed,

$$(a_1+1)(2a_2+1)\cdot...\cdot(ka_k+1)((k+1)a_{k+1}+1) \geq \frac{2^k}{k+1}(1+a_1+2a_2+...+ka_k)((k+1)a_{k+1}+1)$$

Let

$$a=(k+1)a_{k+1}s = a_1+2a_2+...+ka_k \Rightarrow s \geq k.$$

We need to show that

$$\frac{2^k}{k+1}(1+s)(1+a) \geq \frac{2^{k+1}}{k+2}(1+s+a)$$
$$\Leftrightarrow 2(as-k)+k(a-1)(s-1) \geq 0.$$

Since $a \geq 1 \forall k$, the above one is true for $n=k+1$. The proof ends! Equality holds for $a_i = \frac{1}{i}, i=\overline{1,n}$.

Solution 27 (NguyenDungTN). The inequality is equivalent to

$$\left(\frac{1+a_1}{2}\right)\left(\frac{1+2a_2}{2}\right)\cdot ...\left(\frac{1+na_n}{2}\right) \geq \frac{1+a_1+2a_2+...+na_n}{n+1}.$$

Let $x_i = \frac{ia_i-1}{2} \geq 0$, it becomes

$$(1+x_1)(1+x_2)...(1+x_n) \geq 1+\frac{2}{n+1}(x_1+x_2+...+x_n).$$

But

$$(1+x_1)(1+x_2)...(1+x_n) \geq 1+x_1+x_2+...+x_n \geq 1+\frac{2}{n+1}(x_1+x_2+...+x_n).$$

So we have the desired result.

∇

Problem 19 (23, Moldova Team Selection Test 2007). *Let* $a_1, a_2, ..., a_n \in [0,1]$. *Denote* $S = a_1^3 + a_2^3 + ... + a_n^3$. *Prove that*

$$\frac{a_1}{2n+1+S-a_1^3} + \frac{a_2}{2n+1+S-a_2^3} + \cdots + \frac{a_n}{2n+1+S-a_n^3} \le \frac{1}{3}.$$

Solution 28 (NguyenDungTN). By AM-GM Inequality, we have

$$S - a_1^3 + 2(n-1) = (a_2^3+2) + (a_3^3+2) + \cdots + (a_n^3+2) \ge 3(a_2 + a_3 + \cdots + a_n).$$

Thus

$$\frac{a_1}{2n+1+S-a_1^3} \le \frac{a_1}{3(1+a_1+a_2+\cdots+a_n)} \le \frac{a_1}{3(a_1+a_2+\cdots+a_n)}.$$

Similar for $a_2, a_3, ..., a_n$, we have

$$\frac{a_1}{2n+1+S-a_1^3} + \frac{a_2}{2n+1+S-a_2^3} + \cdots + \frac{a_n}{2n+1+S-a_n^3}$$

$$\le \frac{1}{3} \cdot \frac{a_1+a_2+\cdots+a_n}{a_1+a_2+\cdots+a_n} = \frac{1}{3}.$$

The equality holds for $a_1 = a_2 = ... = a_n = 1$.

∇

Problem 20 (24, Peru Team Selection Test 2007). *Let* a, b, c *be positive real numbers, such that*

$$a+b+c \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove that:

$$a+b+c \ge \frac{3}{a+b+c} + \frac{2}{abc}.$$

Solution 29 (NguyenDungTN). By Cauchy-Schwarz Inequality, we have

$$a+b+c \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{9}{a+b+c} \Rightarrow a+b+c \ge 3.$$

Our inequality is equivalent to

$$(a+b+c)^2 \ge 3 + 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right).$$

By AM-GM Inequality

$$2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \le \frac{2}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \le \frac{2}{3}(a+b+c)^2$$

So it is enough to prove that

$$(a+b+c)^2 \geq 3 + \frac{2}{3}(a+b+c)^2 \Leftrightarrow (a+b+c)^2 \geq 9.$$

This inequality is true due to $a+b+c \geq 3$.

Solution 30 (2, DDucLam). We have

$$a+b+c \geq \frac{2}{3}(a+b+c) + \frac{1}{3}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq \frac{2}{3}(a+b+c) + \frac{3}{a+b+c}.$$

We only need to prove that

$$a+b+c \geq \frac{3}{abc},$$

but this inequality is always true since

$$(a+b+c)^2 \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \frac{3}{abc}(a+b+c).$$

∇

Problem 21 (25, Revised by NguyenDungTN). *Let a, b and c be sides of a triangle. Prove that*

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3.$$

Solution 31. Let

$$x = \sqrt{b}+\sqrt{c}-\sqrt{a}, y = \sqrt{c}+\sqrt{a}-\sqrt{b}, z = \sqrt{a}+\sqrt{b}-\sqrt{c},$$

then

$$b+c-a = x^2 - \frac{(x-y)(x-z)}{2}$$

By AM-GM inequality, we have

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} = \sqrt{1-\frac{(x-y)(x-z)}{2x^2}} \leq 1 - \frac{(x-y)(x-z)}{4x^2}$$

We will prove that

$$x^{-2}(x-y)(x-z) + y^{-2}(y-z)(y-z) + z^{-2}(z-x)(z-y) \geq 0.$$

But this immediately follows the general Schur inequality, with the assumption that

$$x \geq y \geq z \Rightarrow x^{-2} \leq y^{-2} \leq z^{-2}.$$

We are done!

∇

Problem 22 (26, Romania Team Selection Tests 2007). *If* $a_1, a_2, \ldots, a_n \geq 0$ *are such that* $a_1^2 + \cdots + a_n^2 = 1$, *find the maximum value of the product* $(1-a_1)\cdots(1-a_n)$.

Solution 32 (hungkhtn, reposted by NguyenDungTN). We use contradiction method. Assume that $x_1, x_2, ..., x_n \in [0,1]$ such that $x_1x_2...x_n = (1-\frac{1}{\sqrt{2}})^2$. We will prove

$$f(x_1, x_2, ..., x_n) = (1-x_1)^2 + (1-x_2)^2 + ... + (1-x_n)^2 \leq 1 \quad (1)$$

Indeed, first, we prove that:

Lemma: If $x, y \in [0,1], x + y + xy \geq 1$ then

$$(1-x)^2 + (1-y)^2 \leq (1-xy)^2.$$

Proof. Notice that

$$(1-x)^2 + (1-y)^2 - (1-xy)^2 = (x+y-1)^2 - x^2y^2$$

$$= (x-1)(y-1)(x+y+xy-1) \leq 0.$$

The lemma is asserted. Return to the problem, let $k = 1 - \frac{1}{\sqrt{2}}$. Assume that $x_1 \leq x_2 \leq ... \leq x_n$, then

$$x_1x_2x_3 \geq k^2 \Rightarrow x_2x_3 \geq k^{4/3},$$

thus

$$x_2 + x_3 + x_2 + x_3 \geq 2k^{2/3} + k^{4/3} = 1.07 \geq 1.$$

Similarly, we have

$$f(x_1, x_2, ..., x_n) \leq f(x_1, x_2x_3, 1, x_4, ..., x_n)$$

$$\leq f(x_1, x_2x_3x_4, 1, 1, x_5, ..., x_n) \leq ... \leq f(x_1, x_2x_3...x_n, 1, 1, ..., 1),$$

From this, easy to get the final result.

∇

Problem 23 (28, Ukraine Mathematic Festival 2007). *Let* $a, b, c > 0$ *và* $abc \geq 1$. *Prove that*

$$a).\ \left(a + \frac{1}{a+1}\right)\left(b + \frac{1}{b+1}\right)\left(c + \frac{1}{c+1}\right) \geq \frac{27}{8}.$$

$$b).\ 27(a^3+a^2+a+1)(b^3+b^2+b+1)(c^3+c^2+c+1)$$
$$\geq 64(a^2+a+1)(b^2+b+1)(c^2+c+1).$$

Solution 33 (pi3.14). Consider the case $abc = 1$. Let $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$. The inequality becomes

$$\sum \frac{\frac{x^2}{y^2} + \frac{x}{y} + 1}{\frac{x}{y} + 1} \geq \frac{27}{8}$$

or

$$8(x^2 + xy + y^2)(y^2 + yz + z^2)(x^2 + zx + z^2) \geq 27xyz(x + y)(y + z)(z + x) \quad (1)$$

We have

$$2(x^2 + xy + y^2) \geq 3\sqrt{xy}(x + y),$$

since

$$2(x^2 + xy + y^2) \geq \frac{3}{2}(x^2 + 2xy + y^2) \geq 3\sqrt{xy}(x + y).$$

Write two similar inequalities, then multiply all of them, we get (1) immediately.

If $abc > 1$, we let $a = ka'$; $b = kb'$; $c = kc'$; with $k = \sqrt[3]{abc}$. We have $k > 1$ and $a'b'c' = 1$. Then

$$\frac{a^2 + a + 1}{a + 1} \geq \frac{a'^2 + a' + 1}{a' + 1}.$$

Since the inequality is proved for a', b', c', this is true for a, b, c immediately.

b). By AM-GM inequality

$$a^2 + 2 \geq 2a \ \Rightarrow (a^2 + 1) \geq \frac{2}{3}(a^2 + a + 1).$$

Therefore

$$3(a^3 + a^2 + a + 1) = 3(a + 1)(a^2 + 1) \geq 6\sqrt{a}.\frac{2}{3}(a^2 + a + 1) = 4\sqrt{a}(a^2 + a + 1).$$

Constructing similar inequalities, then multiply all of them, we get

$$27(a^3+a^2+a+1)(b^3+b^2+b+1)(c^3+c^2+c+1) \geq 64(a^2+a+1)(b^2+b+1)(c^2+c+1).$$

Solution 34 (2, NguyenDungTN). By AM-GM inequality

$$\frac{a + 1}{4} + \frac{1}{a + 1} \geq 1 \ ;$$

$$\frac{3a}{4} + \frac{3}{4} \geq \frac{3}{2}\sqrt{a} \ ;$$

Adding up two inequalities, we get

$$a + \frac{1}{a + 1} \geq \frac{3}{2}\sqrt{a}.$$

Similar for b, c, and finally we have

$$\left(a + \frac{1}{a + 1}\right)\left(b + \frac{1}{b + 1}\right)\left(c + \frac{1}{c + 1}\right) \geq \frac{27}{8}\sqrt{abc} \geq \frac{27}{8}.$$

Equality holds for $a = b = c = 1$.

∇

Problem 24 (29, Asian Pacific Mathematical Olympiad 2007). *Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$. Prove that*

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\geq 1.$$

Solution 35 (NguyenDungTN). We have the transformation

$$\sum_{cyc}\frac{x^2+yz}{\sqrt{2x^2(y+z)}}=\sum_{cyc}\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\sum_{cyc}\sqrt{\frac{y+z}{2}}.$$

Moreover, by Cauchy-Schwarz Inequality

$$\sum_{cyc}\sqrt{\frac{y+z}{2}}\geq\sum_{cyc}\frac{\sqrt{y}+\sqrt{z}}{2}=1.$$

So it is enough to prove that

$$\sum_{cyc}\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}\geq 0$$

Without loss of generality, assume that $x\geq y\geq z$, then

$$\frac{1}{\sqrt{2x^2(y+z)}}\leq\frac{1}{\sqrt{2y^2(z+x)}}\leq\frac{1}{\sqrt{2z^2(x+y)}}.$$

Using the general Schur Inequality, we have the desired result.

∇

Problem 25 (30, Brazilian Olympiad Revenge 2007). *Let $a, b, c\in\mathbb{R}$ with $abc=1$. Prove that*

$$a^2+b^2+c^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 6+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

Solution 36 (NguyenDungTN). Since $abc=1$, we have

$$a^2+b^2+c^2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=a^2+b^2+c^2+2(ab+bc+ca)=(a+b+c)^2.$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2(a+b+c)=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)=(ab+bc+ca)^2.$$

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}+3\right)=\frac{2\left(ab(a+b)+bc(b+c)+ca(c+a)+3abc\right)}{abc}$$

$$= 2(a+b+c)(ab+bc+ca).$$

By AM-GM Inequality,

$$(a+b+c)^2 + (ab+bc+ca)^2 \geq 2|(a+b+c)(ab+bc+ca)| \geq 2(a+b+c)(ab+bc+ca).$$

This ends the proof. The equality holds for $a = b = c = 1$.

∇

Problem 26 (31, Revised by NguyenDungTN). *If x, y, z are positive real numbers, prove that*

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

Solution 37. Using the inequality

$$4(a^2+b^2+ab) \geq 3(a+b)^2 \ \forall a, b (\Leftrightarrow (a-b)^2 \geq 0)$$

We have

$$3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2) \geq \frac{4^3}{3^2}(x+y)^2(y+z)^2(z+x)^2.$$

By AM-GM inequality, we get

$$9(x+y)(y+z)(z+x) = 9(xy(x+y)+yz(y+z)+zx(z+x)+2xyz)$$
$$= 8(xy(x+y)+yz(y+z)+zx(z+x)+3xyz)+xy(x+y)+yz(y+z)+zx(z+x)-6xyz$$
$$\geq 8(x+y+z)(xy+yz+zx).$$

So we have the desired result.

∇

Problem 27 (32, British National Mathematical Olympiad 2007). *Show that for all positive reals a, b, c*

$$(a^2+b^2)^2 \geq (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

Solution 38 (NguyenDungTN). Using the familiar inequality

$$xy \leq \frac{(x+y)^2}{4} \forall x, y \in \mathbb{R},$$

we have

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b) = \left((a+b)^2-c^2\right)\left(c^2-(a-b)^2\right)$$
$$\leq \frac{\left((a+b)^2-c^2+c^2-(a-b)^2\right)^2}{4} = (a^2+b^2)^2.$$

Equality holds when $(a+b)^2 - c^2 = c^2 - (a-b)^2 \ \Leftrightarrow \ c^2 = a^2+b^2$.

∇

Problem 28 (34, Mathlinks, Revised by VanDHKH). *Let* a, b, c, d *be real numbers such that* $a^2 \le 1, a^2+b^2 \le 5, a^2+b^2+c^2 \le 14, a^2+b^2+c^2+d^2 \le 30$ *Prove that* $a+b+c+d \le 10.$

Solution 39. By hypothesis, we have

$$12a^2 + 6b^2 + 4c^2 + 3d^2 \le 120.$$

By Cauchy-Schwarz Inequality, we have

$$100 = (12a^2 + 6b^2 + 4c^2 + 3d^2)\left(\frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{3}\right) \ge (a + b + c + d)^2$$

Therefore $a + b + c + d \le |a + b + c + d| \le 10.$

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