Some inequalities solved by SOS method

Chapter 1

Problems

Problem 1. Prove that for all positive real numbers a, b, c we always have

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4(xy+yz+zx)}$$

(Iran TST 1996)

Problem 2. Suppose that x, y, z are positive real numbers and $xyz \geq 1$. Prove the following inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

(IMO 2005 Pro. A3)

Problem 3. Prove that for all positive real numbers a, b, c we always have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{abc}{2(a^3+b^3+c^3)} \geq \frac{5}{3}$$

(Pham Kim Hung)

Problem 4. Prove that for all positive real numbers a, b, c we always have

$$\frac{2(a^3+b^3+c^3)}{abc} + \frac{9(a+b+c)^2}{a^2+b^2+c^2} \geq 33$$

(Hojoo Lee)

Problem 5. For all positive real numbers a, b, c , prove that

$$\frac{a^4+b^4+c^4}{ab+bc+ca} + \frac{3abc}{a+b+c} \geq \frac{2}{3}(a^2+b^2+c^2)$$

Problem 6. Find the best positive constant k for the inequality

$$\frac{a^3+b^3+c^3}{(a+b)(b+c)(c+a)}+\frac{k(ab+bc+ca)}{(a+b+c)^2}\geq\frac{3}{8}+\frac{k}{3}$$

in which a,b,c are non-negative real numbers.
Problem 7. Find the best positive constant k for the inequality

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{k(a+b)(b+c)(c+a)}{abc}\geq 1+8k$$

with $a,b,c\geq 0$.

Problem 8. Consider the following inequality

$$\frac{2a}{b^3+c^3}+\frac{2b}{c^3+a^3}+\frac{2c}{a^3+b^3}\leq\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

a). Prove that it's not true for all non-negative real numbers a,b,c . b). Prove that it's true if a,b,c are three lengths of an acute triangle.

Problem 9. Let a,b,c be the length of tree sides in a triangle. Prove that:

$$\frac{3(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2$$

Problem 10. Prove that for all nonnegative numbers a,b,c we always have

$$\frac{(a+b)(b+c)(c+a)}{8abc}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2$$

Problem 11. Prove that for all $a,b,c\geq 0$

$$\left(a+\frac{b^2}{c}\right)^2+\left(b+\frac{c^2}{a}\right)^2+\left(c+\frac{a^2}{b}\right)^2\geq\frac{12(a^3+b^3+c^3)}{a+b+c}$$

Problem 12. Prove that for all $a,b,c\geq 0$

$$\frac{a^4}{b^3+c^3}+\frac{b^4}{c^3+a^3}+\frac{c^4}{a^3+b^3}\geq\frac{a+b+c}{2}$$

(Pham Kim Hung)

Problem 13. Prove that for all $a,b,c\geq 0$

$$\frac{a^3}{a^2+2b^2}+\frac{b^3}{b^2+2c^2}+\frac{c^3}{c^2+2a^2}\geq\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2}\geq\frac{a+b+c}{3}$$

(Pham Kim Hung-Tran Le Bach)

Problem 14. Prove that for all $a, b, c \geq 0$

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

Problem 15. Let x, y, z be nonnegative real numbers, find the greatest possible positive value of k such that

$$(x+y+z)^5 \geq k(x^2+y^2+z^2)(x^2y+y^2z+z^2x)$$

The following is also nice

$$(x+y+z)^5 \geq 8(x^2+y^2+z^2)\left[xy(x+y)+yz(y+z)+zx(z+x)\right]$$

(Vasile Cirtoaje)

Problem 16. Let p, R, r denote the half of the perimeter, the radii of the circumcircle and incircle of a triangle. Prove that

$$p^2 \leq 4R^2+4Rr+3r^2$$

Problem 17.

$$\frac{(a-b)(3a+b)}{a^2+b^2}+\frac{(b-c)(3b+c)}{b^2+c^2}+\frac{(c-a)(3c+a)}{c^2+a^2} \geq 0$$

Problem 18.

$$\sum_{\text{Cyclic}} a^4(a-b) \geq 2abc \sum_{\text{Cyclic}} a(a-b)$$

(Vasile Cirtoaje)

Problem 19. ([1])

$$\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}+\frac{a^2+b^2}{c^2+ab} \geq 2+\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Problem 20.

$$\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Problem 21.

$$(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+15 \geq 4\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right)$$

[1] Now, we'll let a, b, c be non-negative real numbers for the following and you need to prove the inequalities bellow

Problem 22.

$$\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq \sqrt{\frac{ab+bc+ca}{3}}$$

(Carlson's Inequality)

Problem 23.

$$a^2+b^2+c^2+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \geq 2(ab+bc+ca)$$

Problem 24. Let a, b, c be the length of tree sides in a triangle. Prove that:

$$\sum_{\text{Cyclic}} \left(\frac{a+b-c}{c}\right)(a^2+b^2) \geq \sum_{\text{Cyclic}} ab\left(\frac{a+b}{c}\right)$$

(Tran Le Bach)

Problem 25.

$$\sum(a+b-c)(b+c-a) \geq \sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$